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ON THE DEGREE OF POLYNOMIAL SUBGROUP GROWTH IN CLASS 2 NILPOTENT GROUPS

BY

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ABSTRACT

We use the theory of zeta functions of groups to establish a lower limit for the degree of polynomial normal subgroup growth in class two nilpotent groups.

1. Introduction

Let G be a finitely generated torsion-free nilpotent group, $\mathfrak{T}$-group in short, and let $s_n^{\triangleleft}(G)$ denote the number of normal subgroups of G of index up to n. Then **the degree of normal polynomial subgroup growth** is

$$\alpha_G^{\triangleleft} = \inf\{\alpha \geq 0 : \text{there exists } c > 0 \text{ with } s_n^{\triangleleft}(G) < cn^{\alpha} \text{ for all } n\}.$$

In this paper we shall give a new lower limit to this quantity using the theory of zeta functions of groups.

In [3] Grunewald, Segal and Smith defined the **normal zeta function** of a $\mathfrak{T}$-group as a formal Dirichlet series of the form

$$\zeta_G^{\triangleleft}(s) = \sum_{H \triangleleft_f G} |G:H|^{-s} = \sum_{n=1}^{\infty} a_n^{\triangleleft}(G) n^{-s},$$

where

$$a_n^{\triangleleft}(G) = |\{H \triangleleft G : |G:H| = n\}|.$$

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The zeta function converges on some right half-plane $\{s \in \mathbb{C} : \Re(s) > \alpha_G^{\triangleleft}\}$, where $\alpha_G^{\triangleleft}$ is the **abscissa of convergence** of $\zeta_G^{\triangleleft}(s)$, which also determines the degree of polynomial subgroup growth.

It is proved in [3] that $\mathfrak{T}$-groups have polynomial subgroup growth.

Proposition 1.1 (GSS): *For a $\mathfrak{T}$-group G,*

$$d = h(G^{ab}) \leq \alpha_G^{\triangleleft} \leq \alpha_G^{\leq} \leq h(G),$$

*where $h(G)$ is the Hirsch length of the group G, and $\alpha_G^{\leq}$ is the degree of polynomial subgroup growth of all the subgroups of G.**

In [2] du Sautoy and Grunewald prove that $\alpha_G^{\triangleleft}$ and $\alpha_G^{\leq}$ are always rational numbers.

For $\mathfrak{T}$-groups of nilpotency class 2, called $\mathfrak{T}_2$-groups, we have a sharper estimate for the abscissa of convergence of the normal zeta functions proved in [3].

Proposition 1.2 (GSS): *Let G be a $\mathfrak{T}_2$-group. Put $h = h(G)$, $d = h(G^{ab})$, $m = h(Z(G))$ and $r = h(G/Z(G))$. Then*

$$\frac{1}{2}(m + r^{-1}) \leq \alpha_G^{\triangleleft} \leq \max\{d, h(1 - r^{-1})\}.$$

The purpose of this paper is to improve the lower limit. In particular, we prove the following.

Theorem 1.3: *Let G be a $\mathfrak{T}_2$-group. Put $h = h(G)$, $d = h(G^{ab})$, $m = h(Z(G))$ and $r = h(G/Z(G))$. Then $\alpha_G^{\triangleleft}$ is at least*

$$\max_{1 \leq k \leq m} \left\{d, \frac{k(m + d - k) + 1}{r + k}\right\}. \tag{1}$$

It is easy to see that this improves the limit given in [3] as soon as the centre has dimension bigger than twice the Hirsch length of the abelianisation.

Example 1.4 ([6])*:* Let $F_{2,d}$ be the free class two nilpotent group on d generators. Then d is also the Hirsch length of the abelianisation and $m = h(Z(F_{2,d})) = h([F_{2,d}, F_{2,d}]) = d(d-1)/2$. Using calculus to estimate the maximum in (1), the abscissa of convergence is approximately at least

$$d(3 - \sqrt{6 + 2d}) + d(d-1)/2.$$

* The Hirsch length of a group G is the number of infinite cyclic factors in a subnormal series with cyclic factors of G.

This is $m + O(d\sqrt{d})$ when $d \to \infty$ and thus improves the previous limit considerably, as the lower limit given in [3] is asymptotically $\frac{1}{2}m + O(d^{-1})$.

Apart from the above, we have very little knowledge of the actual value for the abscissa of convergence for $\mathfrak{T}$-groups in general. There are some specific results in [6] for certain classes of groups, including the normal zeta function of the direct product of n copies of the Heisenberg group and maximal class two quotients of $U_n(\mathbb{Z})$, the group of upper triangular unipotent matrices over $\mathbb{Z}$. In both of these cases, the abscissa of convergence of the normal zeta function is the Hirsch length of the abelianisation.

For subgroup growth Dan Segal has proved that $(3 - 2\sqrt{2})h - \frac{1}{2} \leq \alpha_G^{\leq}$ using pro-p group methods. This result was extended by Klopsch to soluble groups [4].

2. Results

Since zeta functions of groups are Dirichlet series, we can use the following analytic observations. Let (a_n) and (b_n) be sequences of non-negative integers, and let α and β be the respective abscissae of convergence of the Dirichlet series $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$. Then

$$\forall n \quad a_n \leq b_n \Rightarrow \alpha \leq \beta. \tag{2}$$

We shall also use the fact that if $\sum a_n n^{-\gamma}$ is convergent for some real number γ then $\alpha \leq \gamma$.

The Theorem 1.3 is based on the following useful lemma proved in [3].

Lemma 2.1 (GSS): *Let G be a $\mathfrak{T}$-group and A a subgroup of $Z(G)$ such that $G/A \cong \mathbb{Z}^d$. For each $B \leq_f A$ put $X(B)/B = Z(G/B)$. Then*

$$\zeta_G^{\triangleleft}(s) = \zeta_{\mathbb{Z}^d}(s) \cdot \sum_{B \leq_f A} |A : B|^{d-s} |G : X(B)|^{-s}.$$

One can also use the above lemma locally, that is, to count only subgroups of p-power index, since when working with $\mathfrak{T}$-groups, the normal zeta function decomposes as an **Euler product of local factors** over primes p

$$\zeta_G^{\triangleleft}(s) = \prod_p \zeta_{G,p}^{\triangleleft}(s)$$

where

$$\zeta_{G,p}^{\triangleleft}(s) = \sum_{n=0}^{\infty} a_{p^n}^{\triangleleft}(G) p^{-ns}.$$

For $\mathfrak{T}_2$-groups, the centre $Z = Z(G) \cong \mathbb{Z}^m$, where $m = h(Z(G))$. If B is a subgroup of Z of finite p-power index, denoted by $B \leq_p Z$, and gives rise to a quotient group

$$Z/B \cong \mathbb{Z}/p^{n_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{n_k}\mathbb{Z},$$

we say that B is of type $(\underbrace{1, \ldots, 1, p^{n_1}, \ldots, p^{n_k}}_{m})$.

We want to count how many subgroups $B \leq_p Z$ are there that give isomorphic quotient groups, i.e., how many groups belong to the same isomorphism type.

We shall work additively and consider the lattice given by a finite index subgroup in $\mathbb{Z}_p^m$, as there is no harm if we work over $\mathbb{Z}_p$. Now $\mathrm{GL}_m(\mathbb{Z}_p)$ acts transitively on sublattices $B \leq \mathbb{Z}_p^m$ with a quotient $\mathbb{Z}_p^m/B$ of a fixed given isomorphism type

$$(p^{r_0}, p^{r_0+r_1}, p^{r_0+r_1+r_2}, \ldots, p^{r_0+r_1+r_2+\cdots+r_{m-1}}) \\ = p^{r_0}(1, p^{r_1}, p^{r_1+r_2}, \ldots, p^{r_1+r_2+\cdots+r_{m-1}}),$$

where $r_i \geq 0$, relative to the lattice $\mathbb{Z}_p^m$. We call $I := \{i : r_i > 0\}$ the flag type of the lattice.

Note that the index of the lattice of flag type I is

$$|\mathbb{Z}_p^m : B| = p^{\sum_{i \in I}(m-i)r_i}. \tag{3}$$

Let G_B denote the stabiliser of the lattice of the flag type $I = \{i_1, \ldots, i_j\}$. The stabiliser consists of matrices of the form

$$\left(\begin{array}{c|c|c|c|c}
\mathrm{GL}_{i_1}(\mathbb{Z}_p) & p^{r_{i_1}}\mathbb{Z}_p & p^{r_{i_1}+r_{i_2}}\mathbb{Z}_p & \cdots & p^{r_{i_1}+\cdots r_{i_j}}\mathbb{Z}_p \\ \hline
 & \mathrm{GL}_{i_2-i_1}(\mathbb{Z}_p) & p^{r_{i_2}}\mathbb{Z}_p & \cdots & p^{r_{i_2}+\cdots+r_{i_j}}\mathbb{Z}_p \\ \hline
 & & \mathrm{GL}_{i_3-i_2}(\mathbb{Z}_p) & & p^{r_{i_3}+\cdots+r_{i_j}}\mathbb{Z}_p \\ \hline
 & & & \ddots & \vdots \\ \hline
 & & & & \mathrm{GL}_{m-i_j}(\mathbb{Z}_p)
\end{array}\right)$$

with zeros below the diagonal blocks. By the orbit-stabiliser theorem the number of lattices (subgroups) of this type, and thus also of the type

$$(p^{r_0}, p^{r_0+r_1}, p^{r_0+r_1+r_2}, \dots, p^{r_0+r_1+r_2+\dots+r_{m-1}}),$$

is $|\operatorname{GL}_m(\mathbb{Z}_p)/G_B|$.

In order to find the index of the stabiliser in $\operatorname{GL}_m(\mathbb{Z}_p)$, we consider the filtration of $\operatorname{GL}_m(\mathbb{Z}_p)$ by the congruence subgroups:

$$\Gamma_i = \{\gamma \in \operatorname{GL}_m(\mathbb{Z}_p) : \gamma \equiv 1_m \bmod p^i\},$$

where $\Gamma_0 = \operatorname{GL}_m(\mathbb{Z}_p)$. Then $G_B \geq \Gamma_k$ for $k \geq r_1 + \dots + r_{m-1}$. We can use the following formula:

$$\begin{aligned} |\operatorname{GL}_m(\mathbb{Z}_p) : G_B| &= \prod_{i=0}^{k} [\Gamma_i : (\Gamma_i \cap G_B)\Gamma_{i+1}] \\ &= [\Gamma_0 : G_B\Gamma_1] \prod_{i=1}^{k} [\Gamma_i : (\Gamma_i \cap G_B)\Gamma_{i+1}] \\ &= f(p)p^{\sum_{i\in I} r_i(m-i)i - \deg f}. \end{aligned}$$

If we write $g(p) = f(p)p^{-\deg f}$, we have that the index is equal to

$$g(p)p^{\sum_{i\in I} r_i(m-i)i},$$

where $g(p) = p^{-g_0} + \dots + 1$ is a polynomial in p^{-1}, which can be expressed in terms of p-binomial coefficients. Moreover, $g(p)$ is always independent of the particular values r_i take.

We shall construct a subgroup in the centre of the group, which gives us the lower bound, and note that this choice of an isomorphism type is in fact the best possible.

Proposition 2.2: *Let G be a $\mathfrak{T}_2$-group. Put $h = h(G)$, $d = h(G^{ab})$, $m = h(Z(G))$ and $r = h(G/Z(G))$. Then $\alpha_G^{\triangleleft}$ is at least*

$$\max_{1\leq k\leq m} \left\{ d, \frac{k(m+d-k)+1}{r+k} \right\}.$$

Proof: The integer d is a lower bound, as in Proposition 1.1.

For the other expression we need to do some work. For each $B \leq_f Z$ put $X(B)/B = Z(G/B)$. Then by Lemma 2.1

$$\zeta_G^{\triangleleft}(s) = \zeta_{\mathbb{Z}^r}(s) \cdot \sum_{B \leq_f Z} |Z : B|^{d-s} |G : X(B)|^{-s} = \prod_p \sum_{n=1}^{\infty} a_{p^n} p^{-ns}.$$

Since we already have considered d as the lower bound, we will ignore the term $\zeta_{\mathbb{Z}^r}(s)$ in subsequent considerations, since the abscissa of convergence of this is $r \leq d$.

We calculate an approximation to the zeta function by concentrating on isomorphism classes of the type

$$(\underbrace{1,1,\ldots,1}_{k_0},\underbrace{p^n,\ldots,p^n}_{k_1}),$$

where $0 \leq k_0, k_1 \leq m$. Call subgroups of this type **pure**.

We note that a subgroup of the pure type as above has index $p^{k_1 n}$ in the centre of the group. There are $p^{k_0 k_1 n} g(p)$ subgroups of this type in the centre.

The subgroup of type $(\underbrace{p^n,\ldots,p^n}_{r})$ in the abelianisation belongs to the centre of G quotiented out by a subgroup B of type $(\underbrace{1,1,\ldots,1}_{k_0},\underbrace{p^n,\ldots,p^n}_{k_1})$. So $X(B) \geq p^n\mathbb{Z}^r$ and thus $|G : X(B)| \leq p^{nr}$.

We define the zeta function

$$\zeta^{pure}_{G,p}(s) = \sum_{B \leq_{pure} Z} |Z : B|^{d-s} |G : X(B)|^{-s}$$

to count only normal subgroups of pure isomorphism type. Moreover, we will approximate $|G : X(B)|$ by p^{nr} for each $B \leq_{pure} Z$ of pure type, as this will not increase the abscissa considered.

Putting everything together, we have

$$\begin{aligned}
\zeta^{pure}_{G,p}(s) &\geq \sum_{n=0}^{\infty} b_{p^n} p^{-ns} \\
&= 1 + g(p) \sum_{n=1}^{\infty} p^{-nrs} p^{nk_0k_1} p^{nk_1(d-s)} \\
&= 1 + g(p) \sum_{n=1}^{\infty} p^{n(k_0k_1 + k_1 d - (r+k_1)s)} \\
&= 1 + g(p) \frac{p^{k_1(k_0+d)-(r+k_1)s}}{1 - p^{k_1(k_0+d)-(r+k_1)s}} =: 1 + f(p),
\end{aligned}$$

which converges for $\Re(s) > \beta_p$, for some β_p.

It is now clear that $\beta_p \leq \alpha^{\triangleleft}_{G,p}$ since $b_{p^n} \leq a_{p^n}$ as we are counting only a thin section of subgroups. So β_p does indeed provide a lower bound for the local abscissa of convergence of $\zeta^{\triangleleft}_{G,p}(s)$.

It is also clear that if we put

$$\zeta_G^{\triangleleft}(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

and let

$$\prod_p \sum_{n=0}^{\infty} b_{p^n} p^{-ns} = \sum_{n=1}^{\infty} b_n n^{-s},$$

then $\forall n\ a_n \geq b_n$ and so $\alpha_G^{\triangleleft} \geq \beta$, where β is the abscissa of convergence of $\sum_{n=1}^{\infty} b_n n^{-s}$. It remains to determine β.

For global abscissa of convergence, we need to know how the Euler product over all prime numbers converges in relation to the local factors. To infer this, we can use the following facts:

(A) An infinite product $\prod_{n\in J}(1+a_n)$ converges absolutely if and only if the corresponding sum $\sum_{n\in J}|a_n|$ converges.

(B) $\sum_p |p^{-s}|$ converges at $s \in \mathbb{C}$ if and only if $\Re(s) > 1$.

We have

$$\sum_{n=0}^{\infty} b_{p^n} p^{-ns} = 1 + f(p),$$

where

$$f(p) = g(p)\frac{p^{k_0k_1+k_1d-(r+k_1)s}}{1-p^{k_0k_1+k_1d-(r+k_1)s}},$$

and $\prod_p 1 + f(p)$ converges if and only if $\sum_p f(p)$ converges using (A).

Now $\sum_p f(p)$ diverges for

$$\Re(s) \leq \frac{k_0k_1+k_1d+1}{r+k_1} := \beta,$$

because

$$\begin{aligned}\sum_p g(p)\frac{p^{k_0k_1+k_1d-(r+k_1)s}}{1-p^{k_0k_1+k_1d-(r+k_1)s}} &\geq \sum_p \frac{p^{k_0k_1+k_1d-(r+k_1)s}}{1-p^{k_0k_1+k_1d-(r+k_1)s}}\\ &= \sum_p \Big(p^{k_0k_1+k_1d-(r+k_1)s}\\ &\qquad + p^{2(k_0k_1+k_1d-(r+k_1)s)} + \cdots\Big)\end{aligned}$$

which diverges at

$$s = \frac{k_0k_1+k_1d+1}{r+k_1} = \beta,$$

using (B).

It is left to find k_0 and k_1 such that this abscissa is maximal. Changing variables so that $k_1 = m - k$, and $k_0 = k$, we want to find the maximum of

$$\frac{k(m+d-k)+1}{r+k},$$

which is what we claimed. ∎

Remark 2.3: This result is asymptotically as good as we would hope for, because asymptotically it grows as fast as the Hirsch length of the group.

One might wonder if the choice of isomorphism types of the form $(\underbrace{1,1,\dots,1}_{k_0},\underbrace{p^n,\dots,p^n}_{k_1})$ is better than any other choice of isomorphism type, or was the previous proposition only an ad hoc construction? As the next remark shows, this choice gives us the best approximation to the lower bound with the knowledge we have at present.

Remark 2.4: Keep the notation as above. Assume that $B \leq_f Z$ is of the mixed isomorphism type

$$(p^{r_0}, p^{r_0+r_1}, p^{r_0+r_1+r_2}, \dots, p^{r_0+r_1+r_2+\dots+r_{m-1}})$$

where $r_i \geq 0$ and $p^k\mathbb{Z}^r \geq X(B)$, so that $|G : X(B)| \leq p^{kr}$, where $k = r_0 + r_1 + r_2 + \dots + r_{m-1}$. If we consider an approximation $\zeta_{G,p}^{mixed}(s)$ to the zeta function by counting over $B \leq_f Z$ as above, with the approximation that $|G : X(B)| \leq p^{kr}$, then the lattices of the pure type $(\underbrace{1,1,\dots,1}_{k_0},\underbrace{p^n,\dots,p^n}_{k_1})$ determine the abscissa of convergence of this new approximation $\zeta_{G,p}^{mixed}(s)$, which coincides with the abscissa of $\zeta_{G,p}^{pure}(s)$ given in Proposition 2.2.

From Proposition 2.2 it is easy to see the following rough limit, when the abscissa starts to be bigger than the Hirsch length of the abelianisation.

Corollary 2.5: *If $m^2 > 4(rd-1)$, then the abscissa of convergence of the normal zeta functions is larger than the Hirsch length of the abelianisation.*

Proof: Based on the proposition above, we have

$$\alpha_G^{\triangleleft} \geq \max_k \left\{ d, \frac{k(m+d-k)+1}{r+k} \right\}$$

where $k = 1, 2, \ldots, m$. From this it is clear that the abscissa is bigger than d if

$$\frac{k(m+d-k)+1}{r+k} > d$$

for some $k = 1, 2, \ldots, m$. Now

$$\frac{k(m+d-k)+1}{r+k} > d$$

for some k if and only if

$$\max_k k(k-m) + 1 > rd.$$

Using calculus the maximum is attained at $k = m/2$. Substituting this in the expression we deduce that the abscissa is bigger than d if $m^2 > 4(rd-1)$. ∎

Remark 2.6: It is clear that the first group satisfying $m^2 > 4(d^2-1)$ is $F_{2,5}$ where $100 > 96$, and so the global abscissa of convergence is at least $5\frac{1}{10}$.

Determining the abscissa explicitly seems to be surprisingly difficult. Even in class two nilpotent groups and their normal zeta functions, we need to take into account the geometry of the Pfaffian hypersurface associated to the bilinear form induced by taking commutators in the group. This geometry can be quite tricky: even a variety like an elliptic curve can be found embedded in a presentation of a $\mathfrak{T}_2$-group as is seen in [1]. Another example of a variety arising in this context is a quadric four-fold in $\mathbb{P}^5$, which is the variety related to the presentation of the free class two nilpotent group on four generators, $F_{2,4}$. The quadric four-fold contains points, lines and planes as linear subspaces and all these separately affect $|F_{2,4} : X(B)|$ as seen in the explicit calculation in [5]. Thus we also need to be able to describe the dimensions of the varieties of linear subspaces lying on the variety associated to the group and be careful with singular points and other quirks in the geometry. The zeta function seems to reflect all these features in its appearance.

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